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Maps of Surface Groups to Finite Groups with No Simple Loops in the Kernel

by

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Abstract: Let F_g denote the closed orientable surface of genus g. What is the least order finite group, G_g, for which there is a homomorphism $\psi : \pi_1(F_g) \to G_g$ so that no nontrivial simple closed curve on F_g represents an element in $\mathrm{Ker}(\psi)$? For the torus it is easily seen that $G_1 = Z_2 \times Z_2$ suffices. We prove here that G_2 is a group of order 32 and that an upper bound for the order of G_g is given by g^{2g+1}. The previously known upper bound was greater than $2^{g2^{2g}}$.

For any compact surface F there exists a finite group G and a homomorphism $\psi\colon \pi_1(F) \to G$ such that no nontrivial element in the kernel of ψ can be represented by a simple closed curve. Such a homomorphism is said to have *nongeometric kernel*. Casson, Gabai, and Skora [5] have each constructed examples of this (see Section 2 for details). The presence of such examples raises a variety of questions relating to the characterization of the finite groups that can occur in this way. This paper addresses the problem of determining the relationship between the genus of F and the order of G. In the case that F is a torus a complete analysis is straightforward. For instance, the natural projection $\psi\colon \pi_1(F) \to H_1(F; Z_2) \cong Z_2 \times Z_2$ has nongeometric kernel.

Our first result concerns the genus 2 closed orientable surface, F_2. Casson's construction yields a group of order 2^{38}. Skora reduced this order considerably by producing a group of order 2^9. In Section 3 a group of order $2^5 = 32$, G_2, is constructed for which there is a homomorphism $\psi_2\colon \pi_1(F_2) \to G_2$ having nongeometric kernel. In Section 4 it is proved that no such example can be constructed using a group of order less than 32.

The example in Section 3 is generalized to construct examples for arbitrary genus surfaces in Section 5. The order of the groups constructed is quite small compared to previously constructed examples. As the examples directly generalize the minimal genus 2 example, there is the possibility that they are minimal as well.

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1 Notation and Conventions

Throughout this paper all surfaces will be closed and orientable. References to basepoints for the fundamental group of a space are omitted. Since the property of being in the kernel of a homomorphism depends only on the conjugacy class of an element, such omissions will not affect the arguments.

By a *simple loop* on a surface we mean an embedding of the circle S^1 .

We will say that a homomorphism $\psi\colon \pi_1(F) \to G$ has *geometric kernel* if some nontrivial element in the kernel can be represented by a simple loop. Otherwise ψ has *nongeometric kernel*.

2 Basic Examples

In this section a procedure of Casson is used to construct for each surface F a finite group G and a surjective homomorphism $\psi : \pi_1(F) \to G$ such that ψ has nongeometric kernel. The orders of the groups involved is computed for contrast with the examples produced in Section 5.

The statement that ψ has nongeometric kernel can be reinterpreted in terms of covering spaces as

follows. Corresponding to Ker(ψ) there is a connected regular covering space $p\colon \tilde{F} \to F$ with $p_*(\pi_1(F)) = \mathrm{Ker}(\psi)$. An element in $\pi_1(F)$ is in Ker(ψ) if and only if when represented by a closed path, the path lifts to a closed path in $\tilde{F}$. Hence a simple loop on F represents an element in Ker(ψ) if and only if it can be lifted to a simple loop in $\tilde{F}$. Conversely, if $p : \tilde{F} \to F$ is a regular covering space with the property that no nontrivial simple loop on F lifts to $\tilde{F}$ then the natural projection $\psi : \pi_1(F) \to \pi_1(F)/p_*(\pi_1(\tilde{F}))$ has nongeometric kernel.

Construction Given a surface F, construct the covering space $p\colon \tilde{F} \to F$ corresponding to the kernel of the projection $\pi_1(F) \to H_1(F; Z_2)$. Simple nonseparating loops on F represent generators of $H_1(F; Z_2)$ and hence do not lift to $\tilde{F}$. Nontrivial separating simple loops do lift, but each preimage on $\tilde{F}$ is nonseparating on $\tilde{F}$.

Now construct the covering $q\colon \bar{F} \to \tilde{F}$ corresponding to the kernel of the projection $\pi_1(\tilde{F}) \to H_1(\tilde{F}; Z_2)$. As no nonseparating simple loop on $\tilde{F}$ lifts to $\bar{F}$ it is apparent that no nontrivial simple loop on F lifts to $\bar{F}$.

It remains to show that the covering $p\circ q\colon \bar{F} \to F$ is regular; that is, that $p_*\circ q_*(\pi_1(\bar{F}))$ is normal in $\pi_1(F)$. Observe that $q_*(\pi_1(\bar{F}))$ is a characteristic subgroup of $\pi_1(\tilde{F})$ and $p_*(\pi_1(\tilde{F}))$ is a characteristic subgroup of $\pi_1(F)$. Since a characteristic subgroup of a characteristic subgroup is characteristic, $p_* \circ q_*(\pi_1(\bar{F}))$ is characteristic in $\pi_1(F)$, and is hence normal.

Order of $\pi_1(F)/ < p_* \circ q_*(\pi_1(\bar{F})) >$

The order of this finite group is equal to the degree of the covering $p \circ q$. Suppose that F is of genus g. The Euler characterisitic of F is $2 - 2g$. Since $\tilde{F}$ is a 2^{2g} fold cover of F, the Euler characteristic of $\tilde{F}$ is $2^{2g}(2 - 2g)$. The genus of $\tilde{F}$ is $\frac{1}{2}(2 - 2^{2g}(2 - 2g)) = \frac{1}{2}((g-1)2^{2g+1} + 2) = g'$. The covering $q\colon \bar{F} \to \tilde{F}$ is of degree $2^{2g'}$. The degree of $p \circ q\colon \bar{F} \to F$ is the product of these two degrees: $2^{2g'}2g = 2^{(g-1)2^{2g+1}+2+2g}$.

Note The construction of Gabai differs considerably from the one above. He notes that every simple curve is in the complement of some index three subgroup of $\pi_1(F)$; nonseparating curves are not in the kernel of some map to Z_3 and separating curves are mapped to a 3–cycle in the third symmetric group, S_3, under some homomorphism and hence map to the complement of an index three subgroup of S_3. Hence the quotient map $\pi_1(F)\longrightarrow\pi_1(F)/H$ has nongeometric kernel, where H is the intersection of all index three subgroups of $\pi_1(F)$. We have been unable to find reasonable bounds on the size of this quotient.

3 A small genus 2 example

The groups constructed in the previous section are of very large order. If F is of genus 2, the corresponding group is of order 2^{38}. This section presents a description of a group of order 32, G_2, and a homomorphism ψ_2 of the fundamental group of the genus 2 surface to G_2, such that ψ_2 has a nongeometric kernel. The next section contains a proof that this example is minimal.

For the remainder of this section F will denote a genus 2 surface.

Construction of G_2 For our purposes, the easiest way to describe G_2 is as follows. Define a group structure on the set $(Z_2)^4 \times Z_2$ by defining the product by

$$(a_1, b_1, a_2, b_2, \epsilon)(a_1', b_1' a_2', b_2', \epsilon') = (a_1 + a_1', b_1 + b_1', b_2 + a_2', b_2 + b_2', \epsilon + \epsilon' + b_1 a_1' + b_2 a_2').$$

The operations within the parenthesis are addition and multiplication in Z_2. The verification that this defines a group structure can be done by a direct calculation, which is left to the reader. The group is denoted 32_{42} in [6], and $\Gamma_5 a_1$ in the notation of [3].

An essential calculation for later purposes is that of commutators in G_2. A direct computation yields

$$[(a_1, b_1, a_2, b_2, \epsilon), (a_1', b_1' a_2', b_2', \epsilon')] = (0, 0, 0, 0, (b_1 a_1' - b_1' a_1) + (b_2 a_2' - b_2' a_2)). \tag{1}$$

From this it is apparent that both the center and commutator subgroup of G_2 consists of the set $(0, 0, 0, 0) \times Z_2$. The abelianization of G_2 is $(Z_2)^4$, given by the projection $(Z_2)^4 \times Z_2 \to (Z_2)^4 \times \{0\}$.

Construction of ψ_2 Let $\{x_1, y_1, x_2, y_2\}$ be a standard generating set of $\pi_1(F)$ so that $\pi_1(F)$ has presentation $< x_1, y_1, x_2, y_2, [x_1, y_1][x_2, y_2] = 1 >$. This set projects to a standard symplectic basis of $H_1(F; Z_2)$, $\{|x_1|, |y_1|, |x_2|, |y_2|\}$

Define $\psi_2 \colon \pi_1(F) \to G_2$ be setting:

$$\psi_2(x_1) = (1, 0, 0, 0) \times (0),$$

$$\psi_2(y_1) = (0, 1, 0, 0) \times (0),$$

$$\psi_2(x_2) = (0, 0, 1, 0) \times (0),$$

$$\psi_2(x_2) = (0, 0, 0, 1) \times (0).$$

Using (1) it is easily verified that this gives a well defined surjective representation.

The key observation is that ψ_2 has the following property: if ω_1 and ω_2 are elements of $\pi_1(F)$, then

$$[\psi_2(\omega_1), \psi_2(\omega_2)] = (0, 0, 0, 0) \times (|\omega_1| \cap |\omega_2|), \tag{2}$$

where $|\omega_1| \cap |\omega_2|$ is the Z_2 intersection number of the classes in $H_1(F; Z_2)$, represented by ω_1 and ω_2. This follows from (1) along with the fact that if (a_1, b_1, a_2, b_2) and (a_1', b_1', a_2', b_2') are classes in $H_1(F; Z_2)$, then $(a_1, b_1, a_2, b_2) \cap (a_1', b_1', a_2', b_2') = (b_1 a_1' - b_1' a_1) + (b_2 a_2' - b_2' a_2)$. (Note also that the natural map $\pi_1(F) \to H_1(F; Z_2)$ factors through G_2 via ψ_2.)

The kernel of ψ_2 is nongeometric Suppose that there is a simple loop γ representing a nontrivial element ω in $\mathrm{Ker}(\psi_2)$. Our first observation is that γ can be chosen to be separating. If γ is nonseparating, pick a simple loop γ' meeting γ transversely in exactly one point. Let ω' be the element of $\pi_1(F)$ represented by γ'. Clearly, $[\omega, \omega']$ is in the kernel of ψ_2 and it is represented by a separating simple loop.

Since γ is now assumed to be separating, it bounds a punctured torus on F. This follows from the classification of surfaces. Hence $\omega = [\omega_1, \omega_2]$, where ω_1 and ω_2 are represented by simple loops meeting transversely in one point. From this one computes using (2) that $\psi_2(w) = [\psi_2(\omega_1), \psi_2(\omega_2)] = (0, 0, 0, 0) \times (|\omega_1| \cap |\omega_2|) = (0, 0, 0, 0) \times (1)$, which is nontrivial in G_2. This contradicts the assumption that $w \in \mathrm{Ker}(\psi_2)$.

4 Minimality of G

The goal of this section is to prove that if F is of genus 2 and the order of G is less than 32, than any homomorphism $\phi \colon \pi_1(F) \to G$ has geometric kernel.

Here is a summary of the argument. We first prove that any $\phi : \pi_1(F) \to G$ has geometric kernel if G is a cyclic extension of an abelian group, that is, if G contains a normal abelian subgroup with cyclic quotient. The approach used to prove this was pointed out by Allan Edmonds. The argument depends on an analysis of the action of the homeomorphism group of F on the set of representations of $\pi_1(F)$ to G. We next note that with the exception of two groups of order 24, $SL_2(Z_3)$ and S_4, all groups of order less than 32 are cyclic extensions of abelian groups. This can be proved by a case–by–case analysis using the Sylow theorems. More easily, group tables such as [6] provide the necessary information. The proof is completed using specialized arguments for $SL_2(Z_3)$ and S_4.

Cyclic Extension of Abelian Groups Fix a group G. The group of basepoint preserving homeomorphisms of F acts on the set of representations of $\pi_1(F)$ to G, as follows. If h is a homeomorphism of F, send a representation ϕ to $\phi \circ h_*$. Notice that ϕ has geometric kernel if and only if $\phi \circ h_*$ has geometric kernel. The following is a result of Nielsen [4]; a proof can be found in [1].

4.1 Lemma. *If G is a cyclic group and $\phi : \pi(F) \to G$ is a surjective homomorphism, then there is a homeomorphism h of F such that $\phi \circ h_*(x_1)$ generates G , and $\phi \circ h_*(y_1)$, $\phi \circ h_*(x_2)$ and $\phi \circ h_*(y_2)$ are all trivial.*

4.2 Theorem. *If G contains an abelian normal subgroup N such that G/N is cyclic, than any surjective homomorphism $\phi\colon \pi_1(F) \to G$ has a geometric kernel.*

Proof Denote the quotient map $G \to G/N$ by ρ. Applying the lemma, we can assume that $\rho \circ \phi(x_2)$ and $\rho \circ \phi(y_2)$ are both trivial. Hence $\phi(x_2)$ and $\rho \circ \phi(y_2)$ are both in N. The commutator $[x_2, y_2]$ is represented by a simple loop and is in the kernel of ϕ, since $\phi([x_2, y_2])$ is in the commutator subgroup of an abelian group.

Exceptional Groups

Case 1 We begin by recalling that $SL_2(Z_3)$ is isomorphic to the semidirect product of the quaternionic 8–group, Q, with Z_3. We will use the standard notation for elements in Q . The generator of Z_3 will be denoted t. The action of Z_3 on Q is given by $tit^{-1} = j$, $tjt^{-1} = k$ and $tkt^{-1} = i$. Note that $t(-1)t^{-1} = -1$ and that $-1 \in Q$ is hence central in $SL_2(Z_3)$.

Suppose $\phi : \pi_1(F) \to SL_2(Z_3)$ is a surjective representation with nongeometric kernel. Applying the lemma to the composition $\pi_1(F) \to SL_2(Z_3) \to SL_2(Z_3)/Q = Z_3$ we can assume that $\phi(x_1) = tq_1$, $\phi(y_1) = q_2$, $\phi(x_2) = q_3$, and $\phi(y_2) = q_4$, where each q_i is in Q.

Since $[x_2, y_2]$ is represented by a simple loop, $[q_3, q_4] \neq 1$. Hence $[q_3, q_4] = -1 \in Q$. It follows that $[tq_1, q_2] = -1$. Note that $q_2 \neq \pm 1$, so $q_2 = \pm i$, $\pm j$, or $\pm k$. From the commutator relation, $tq_1q_2q_1^{-1}t^{-1} = q_2^{-1}$ For any two quaternions, $q_1q_2q_1^{-1} = q_2^{\pm 1}$. Hence, $tq_2t^{-1} = q_2^{\pm 1}$. However, this is impossible, given that $q_2 \neq \pm 1$ and the action of t on Q.

Case 2 The symmetric group S_4 is the semidirect product of $Z_2 \times Z_2$ with S_3. As a subgroup, the $Z_2 \times Z_2$ is given by the set $\{(1), (12)(34), (13)(24), (14)(23)\}$ The S_3 is given by the set $\{(1), (12), (13), (23), (123), (321)\}$.

Let $\phi : \pi_1(F) \to S_4$ be a surjective representation with nongeometric kernel. The main result of [2] applied to the composition $\pi_1(F) \to S_4 \to S_4/(Z_2 \times Z_2) \cong S_3$ shows that by applying a homeomorphism we can arrange that ϕ takes on the values $\phi(x_1) = (12)n_1$, $\phi(y_1) = n_2$, $\phi(x_2) = (123)n_3$, and $\phi(y_2) = n_4$ where each n_i is in $Z_2 \times Z_2$.

Since both y_1 and y_2 are represented by simple loops, neither n_2 nor n_4 are trivial. Also, $[x_1, y_1]$ is represented by a simple loop, so $[(12)n_1, n_2] \neq 1$. It follows that $n_2 \neq (12)(34)$. There are two other possibilities for n_2.

Suppose that $n_2 = (13)(24)$. There are three possible values of n_4 to be considered. Because y_1y_2 is realized by a simple loop, $n_4 \neq (13)(24)$. It is easily seen that $y_1x_2y_2^{-1}x_2^{-1}$ is realized by a simple loop. Hence $n_4 \neq (12)(34)$. Finally $n_4 \neq (14)(23)$, because $y_2x_1y_1^{-1}x_1^{-1}$ can also be represented by a simple loop.

We proceed similarly if $n_2 = (14)(23)$. Clearly $n_4 \neq (14)(23)$. Because $y_2x_1y_1^{-1}x_1^{-1}$ is realized by a simple loop, $n_4 \neq (13)(24)$. Finally, it is again easily seen that $y_1x_2^{-1}y_2^{-1}x_2$ is realized by a simple loop. This implies that $n_4 \neq (12)(34)$. All possibilities have now been eliminated.

5 Generalizations

The group constructed in Section 3, G_2, can be generalized to a group G_k such that for the genus k surface F_k there is a homomorphism $\phi_k\colon \pi_1(F_k) \to G_k$ with nongeometric kernel. The arguments are similar to those of Section 3 and are only outlined here.

Define G_k by defining a product on the set $(Z_k)^{2k} \times Z_k$ as follows.

$$(a_1, b_1, a_2, b_2, \ldots b_k, \epsilon)(a_1', b_1', a_2', b_2', \ldots b_k', \epsilon') = (a_1 + a_1', b_1 + b_1', a_2 + a_2', b_2 + b_2', \ldots, b_k + b_k', \epsilon + \epsilon' + \sum b_i a_i')$$

Sums and products within the parenthesis are in Z_k. That this defines a group is a straightforward calculation.

There is a natural representation $\phi_k\colon \pi_1(F_k) \to G_k$ as before. In this case the essential observation is

$$[\phi_k(\omega_1), \phi_k(\omega_2)] = (0, 0, \ldots, 0) \times (|\omega_1| \cap |\omega_2|), \tag{1}$$

where $(|\omega_1| \cap |\omega_2|)$ is the Z_k intersection number of the classes in $H_1(F_k; Z_k)$ represented by ω_1 and ω_2.

If ϕ_k had geometric kernel, there would be a separating simple loop representing an element in the kernel. Using the classification of surfaces, that element would be of the form

$$[\omega_1, \omega_1'][\omega_2, \omega_2'] \cdots [\omega_m, \omega_m']$$

with $m < k$ and $(|\omega_i| \cap |\omega_i'|) = 1$ for all i. A contradiction follows as in Section 3.

Remark The order of the group G just constructed is g^{2g+1}. This number should be contrasted to the order found in Section 2, $2^{(g-1)2^{2g+1}+2+2g}$. The first is obviously much smaller than the second. The results of this paper, along with our difficulties in trying to find smaller examples, leads us to conjecture that g^{2g+1} represents the least possible order.

References

[1] Edmonds, A. *Surface Symmetry I*, Michigan J. Math 29 (1982) 171–183.
[2] Edmonds, A. *Surface Symmetry II*, Michigan J. Math 30 (1983) 143–154.
[3] Hall, M. and Senior, J.K. *The groups of order* $2^n (n \leq 6)$ Macmillan, New York (1964).
[4] Nielson, J. *Die Struktur periodischer Transformationen von Flachen*, Dansk Vid. Selsk., Mat.-Fys. Medd. 15 (1937), 1–77.
[5] Skora, R. *Dissertation*, Department of Mathematics, University of Texas, Austin, Texas, 1984.
[6] Thomas, A.D. and Wood, G.V. *Group Tables*, Shiva Publishing Limited, Kent, Great Britian, 1980.

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